Section A

[64 marks]

Answer all questions.

1.

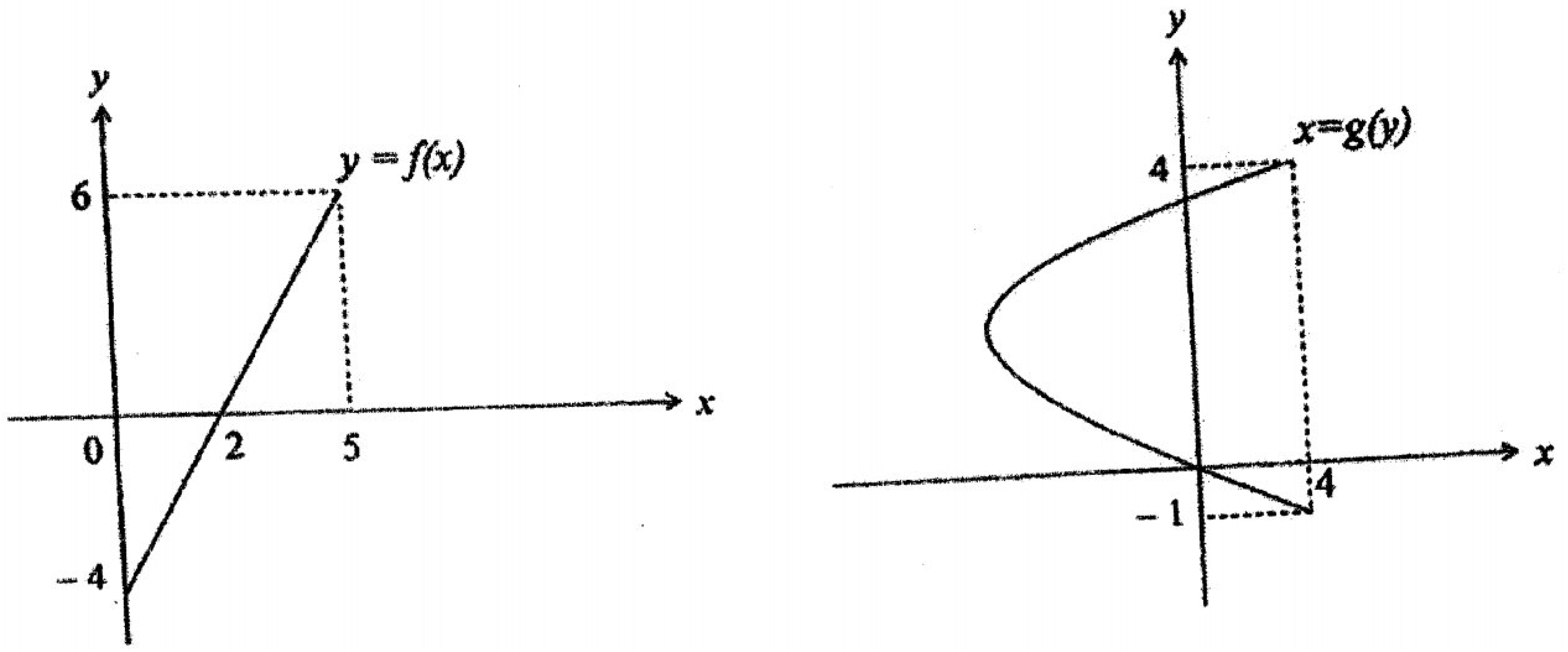


Diagram 1

Diagram 1 shows the graph $y = f(x)$ *and* $x = g(y)$.

a) Determine which graph represents a function and state your reason. [1 mark]

b) Hence, state

i) the domain

ii) the range

iii) the function notation

of the function. [3 marks]

Answer :

(a) graph $y = f(x)$ is a function.

The vertical line test cut one point only

(b) (i) domain $= \{0 \leq x \leq 5\}$.

(ii) range $= \{-4 \leq y \leq 6\}$

(iii) $m = \frac{6-(-4)}{5-0}$

$m = 2$

$c = -4$

$f(x) = 2x - 4$

2. A quadratic function $f(x) = -x^2 + kx + 7$ where k is a constant, has a maximum point (3, m).

a) By using completing the square, find the values of k and m. [3 marks]

b) Express $f(x)$ in the form of $f(x) = a(x-p)(x-q)$, where a, p and q are constants. Hence state the values of p and q. [2 marks]

Answer :

(a) $f(x) = -x^2 + kx + 7$

$f(x) = -(x^2 - kx - 7)$

$f(x) = -[x^2 - kx + (\frac{-k}{2})^2 - (\frac{-k}{2})^2 - 7]$

$f(x) = -[(x - \frac{k}{2})^2 - (\frac{k^2}{4}) - 7]$

$f(x) = -[(x - \frac{k}{2})^2 - (\frac{k^2 - 28}{4})]$

$f(x) = -(x - \frac{k}{2})^2 + \frac{k^2 - 28}{4}$

$x - \frac{k}{2} = 0$, $\frac{k^2 - 28}{4} = m$

$3 = \frac{k}{2}$ $\qquad \frac{6^2 - 28}{4} = m$

$k = 6$ $\qquad m = 2$

(b) $f(x) = -x^2 + kx + 7$

$f(x) = -(x^2 - kx - 7)$

$f(x) = -(x^2 - 6x - 7)$

$f(x) = -(x - 7)(x + 1)$

Compare with

$f(x) = a(x - p)(x - q)$

$\therefore p = 7$, $q = -1$

3. a) Given $g : x \rightarrow 8x - 4$ and $h : x \rightarrow \frac{x}{2} - 6$, find the value of x if $2h(x) = g(x)$. [2 marks]

b) Given that $h^{-1}(x) = \frac{p}{q-x}$, $x \neq q$. If $h(1) = 3$ and $h^{-1}(5) = -1$, find the values of p and q. [3 marks]

Answer :

(a) $g(x) = 8x - 4 \quad h(x) = \frac{x}{2} - 6$

$2h(x) = g(x)$

$2\left(\frac{x}{2} - 6\right) = 8x - 4$

$\cancel{2}\left(\frac{x-12}{\cancel{2}}\right) = 8x - 4$

$x - 12 = 8x - 4$

$7x = -8$

$x = -\frac{8}{7}$

(b) $h^{-1}(x) = \frac{p}{q-x}$

$h^{-1}(5) = -1$

$-1 = \frac{p}{q-5}$

$-1(q-5) = p$

$p = -q + 5 - (1)$

$h^{-1}(x) = \frac{p}{q-x}$

Let $y = \frac{p}{q-x}$

$q - x = \frac{p}{y}$

$x = q - \frac{p}{y}$

$h(x) = q - \frac{p}{x}$

$h(1) = 3$

$q - \frac{p}{1} = 3$

$q - p = 3 - (2)$

sub (1) into (2):

$q - (-q + 5) = 3$

$q + q - 5 = 0$

$2q = 5$

$q = \frac{5}{2}$

sub $q = \frac{5}{2}$ into (1):

$p = -\left(\frac{5}{2}\right) + 5$

$p = \frac{5}{2}$

4. Given $f(x) = \frac{4}{(1-2x)^3}$ $and\ f'(x) = m(1-2x)^n$

a) Find the value of m + n. [3 marks]

b) Hence, find the value of p if $\frac{1}{p}\int f'(x)dx = -\frac{1}{81}$ $and\ x = 2.$ [2 marks]

Answer :

~~(a) $f(x) = \frac{4}{(1-2x)^3}$~~

~~Let $y = \frac{4}{(1-2x)^3}$~~

~~$(1-2x)^3 = \frac{4}{y}$~~

~~$1-2x = \sqrt[3]{\frac{4}{y}}$~~

(a) $f(x) = \frac{4}{(1-2x)^3}$

$f(x) = 4(1-2x)^{-3}$

$f'(x) = \frac{4(-3)(1-2x)^{-3-1}}{(-4)(-2)}$

$f'(x) = \frac{-12(1-2x)^{-4}}{8}$

$f'(x) = -3(1-2x)^{-4}$

$\therefore m = -3,\ n = -4$

(b) $\frac{1}{p}\int f'(x)dx = -\frac{1}{81}$

$\frac{1}{p}\left[\frac{4}{(1-2x^3)}\right] = -\frac{1}{81}$

$\frac{1}{p}\left[\frac{4}{(1-2(2))^3}\right] = -\frac{1}{81}$

$\frac{1}{p}\left(-\frac{4}{27}\right) = -\frac{1}{81}$

$\frac{1}{p} = -\frac{1}{81} \times -\frac{27}{4}$

~~$\frac{1}{p} = \frac{5}{105}$~~ $\frac{1}{p} = \frac{1}{12}$

~~$p = \frac{108}{5}$~~ $p = 12$

5.

X = r	0	1	2	3	4	5	6	7
P(X = r)		$\frac{416}{729}$					$\frac{11}{243}$	

Table 5

Table 5 shows part of a binomial distribution, X ~B(7, p) for 7 trials. The probability of success for each trial is p.

a) Find the value of P(X = 3) + P(X = 4) in fraction form. [2 marks]

b) Hence, show $p = \frac{2}{9q}$ if q is the probability of failure for each trial. [2 marks]

c) If q = 2p, find the values of p and q. [2 marks]

Answer :

(a) $P(X=3)+P(X=4) = 1 - \frac{416}{729} - \frac{11}{243}$

$= \frac{280}{729}$

(b) ${}^7C_3 \times p^3 \times q^4 + {}^7C_4 \times p^4 \times q^3 = \frac{280}{729}$

$35p^3q^4 + 35p^4q^3 = \frac{280}{729}$

$35p^3q^3(q+p) = \frac{280}{729}$

$p+q=1$

$35p^3q^3(1) = \frac{280}{729}$

$p^3q^3 = \frac{280}{729} \div 35$

$p^3q^3 = \frac{8}{729}$

$(pq)^3 = \frac{8}{729}$

$pq = \sqrt[3]{\frac{8}{729}}$

$pq = \frac{2}{9}$

$p = \frac{2}{9q}$ (shown)

(c) [illegible]

$p = \frac{2}{9(2p)}$

$p = \frac{2}{18p}$

$p = \frac{1}{3}$

$q = 2p$

$= 2\left(\frac{1}{3}\right)$

$= \frac{2}{3}$

6.

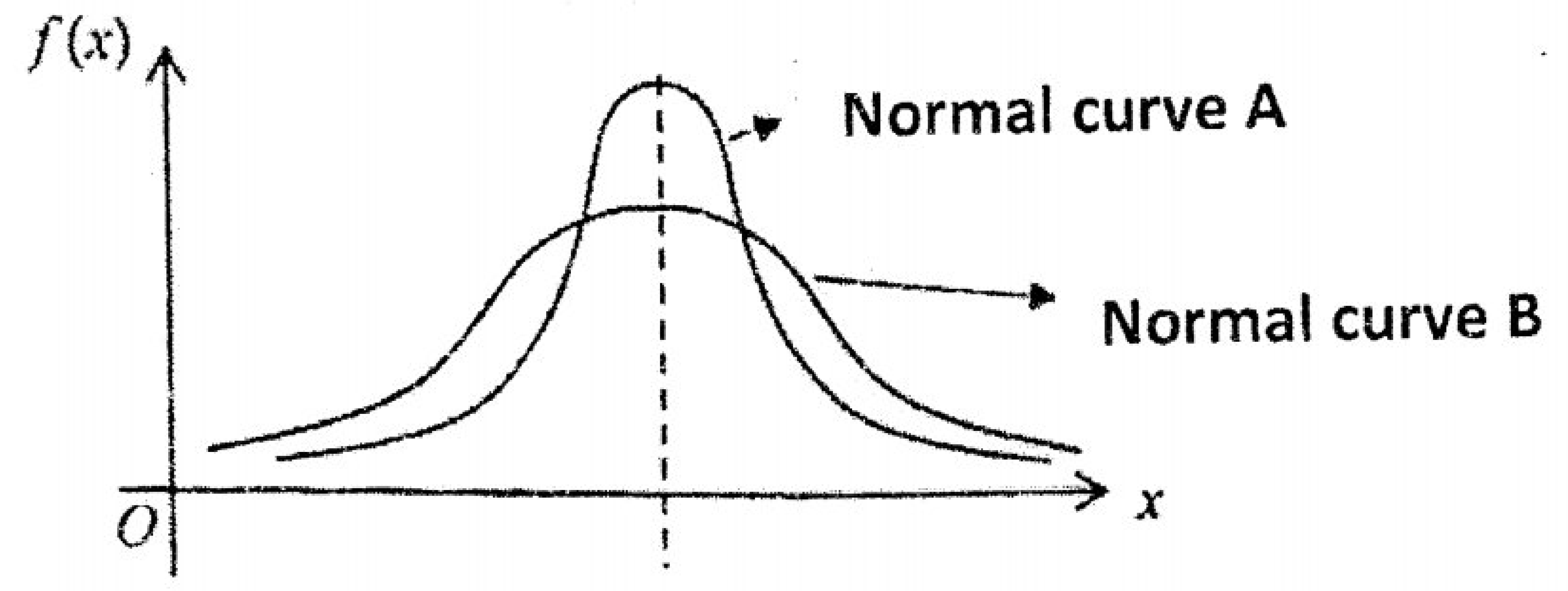


Diagram 6

a) Diagram 6 shows two normal curves a and b with $\mu = 500$. If the variance of data X and data Y are 1200 and 3600 respectively, determine the appropriate curve representing data X and data Y. [2 marks]

b) The mass of the chicken produced by a farm follows a normal distribution with a mean of 2.4 kg and a standard deviation of m kg. It is given that the mass of 16.35% of the chicken are more than 2.8 kg.

i) Find the value of m. [2 marks]

ii) If there are 1580 chickens ready for the market, find the number of chickens with the mass between 2.0 kg and 2.6 kg. [3 marks]

Answer :

a)

Data X	Normal curve A
Data Y	Normal curve B

b) (i) $\mu = 2.4$

$\sigma = m$

$P(x > 2.8) = 0.1635$

$P\left(Z > \frac{2.8 - 2.4}{m}\right) = 0.1635$

$\frac{0.4}{m} = 0.1635$

$m = 2.446$

(ii) $P(2.0 < x < 2.6)$

$= P\left(\frac{2.0 - 2.4}{2.446} < Z < \frac{2.6 - 2.4}{2.446}\right)$

$= P(-0.164 < Z < 0.082)$

$= 1 - P(Z > -0.164) - P(Z > 0.082)$

$= 1 - P(Z < 0.164) - P(Z > 0.082)$

$= 1 - \cancel{0.98}\ (0.4364 - 0.0016)$

$- (0.4681 - 0.0008)$

$= 0.0931$

7.

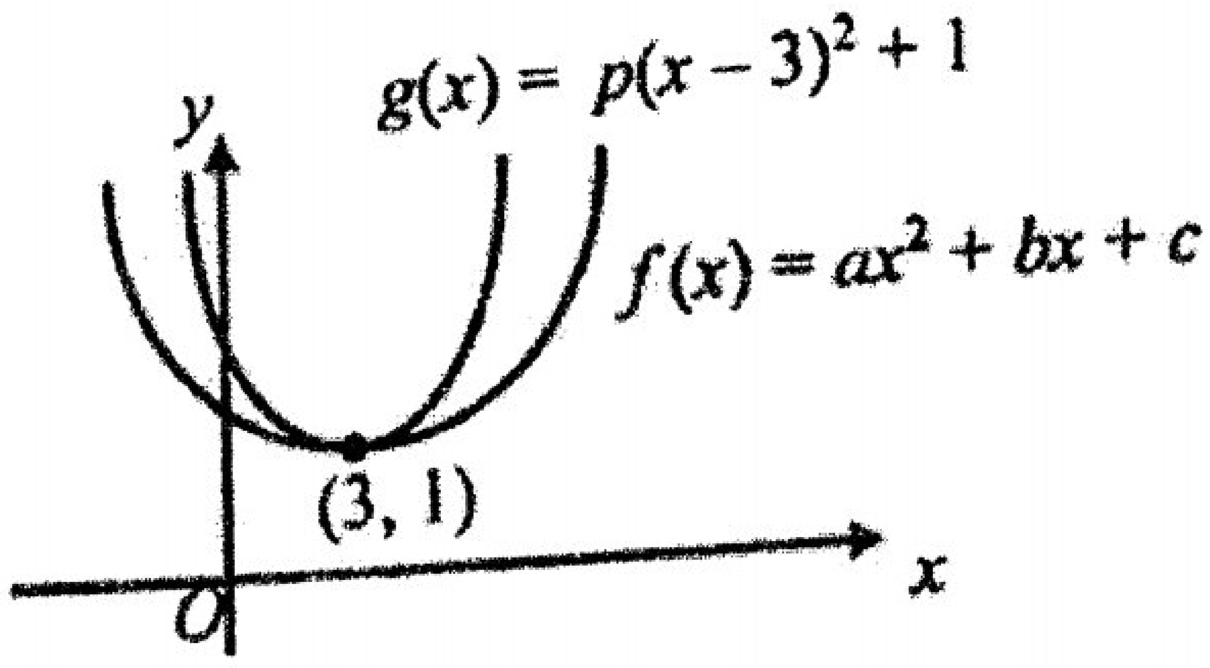


Diagram 7

a) Diagram 7 shows the graphs of two quadratic functions $f(x) = ax^2 + bx + c$ and $g(x) = p(x-3)^2 + 1$.

i) State the range of a in terms of p. [1 mark]

ii) State the new quadratic function of g(x) if the graph g(x) moves 5 units to the left. [1 mark]

b) Find the values of k such that $(4-k)x^2 - 2(k-3)x + k = 3$ has two equal roots. [3 marks]

Answer :

a(i) $a > p$

(ii)

$g(x) = p(x+2)^2 + 1$

(b) $(4-k)x^2 - 2(k-3)x + k = 3$

$(4-k)x^2 - 2(k-3)x + k - 3 = 0$

$b^2 - 4ac = 0$

$[-2(k-3)]^2 - 4(4-k)(k-3) = 0$

$(-2k+6)^2 - 4(4k - 12 - k^2 + 3k) = 0$

$4k^2 - 24k + 36 - 16k + 48 + 4k^2 - 12k = 0$

$\cancel{8k^2 - 40k + 57 = 0}$

$8k^2 - 52k + 84 = 0$

8.

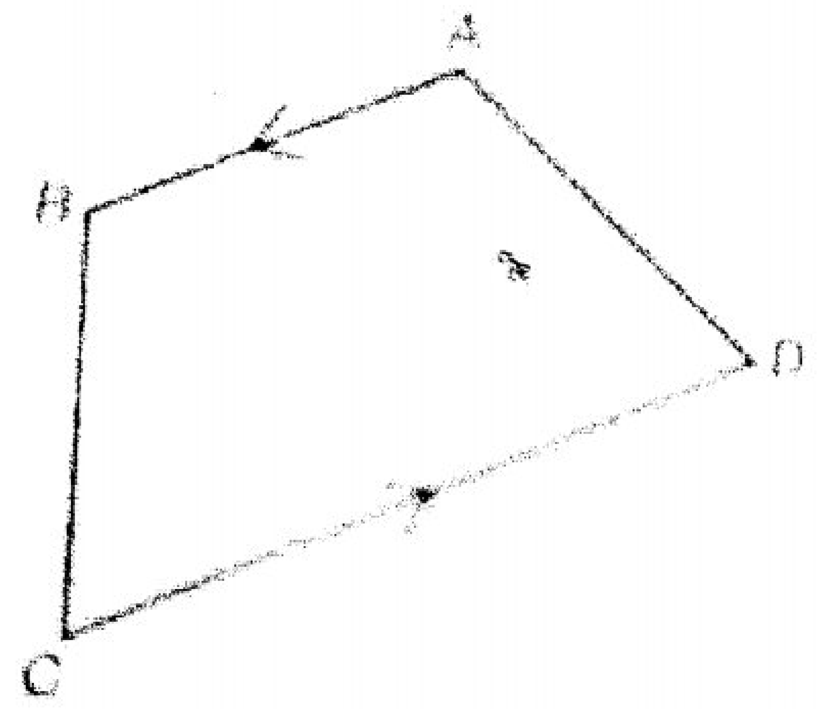


Diagram 8

Diagram 8 shows a trapezium ABCD.

Given that $\overrightarrow{AB} = (h+2)\underset{\sim}{i} - 3\underset{\sim}{j}$, $\overrightarrow{CD} = 2\underset{\sim}{i} + 3k\underset{\sim}{j}$ *and* $2\overrightarrow{AB} = \overrightarrow{DC}$, find

a) the values of h and of k. [3 marks]

b) the unit vector in the direction $\overrightarrow{AB}$. [2 marks]

Answer :

(a) $2\overrightarrow{AB} = \overrightarrow{DC}$

$2[(h+2)\underset{\sim}{i} - 3\underset{\sim}{j}] = -(2\underset{\sim}{i} + 3k\underset{\sim}{j})$

$(2h+4)\underset{\sim}{i} - 6\underset{\sim}{j} = -2\underset{\sim}{i} - 3k\underset{\sim}{j}$

$2h+4 = -2$, $-6 = -3k$

$2h = 2$ $\qquad k = \frac{-6}{-3}$

$h = 1$ $\qquad k = 2$

$\therefore h = 1$, $k = 2$

~~$\hat{r} = \sqrt{(1+2)^2 + (-3)^2}$~~

(b) $\hat{\underset{\sim}{r}} = \dfrac{\sqrt{(1+2)^2 + (-3)^2}}{(1+2)\underset{\sim}{i} - 3\underset{\sim}{j}}$

$\hat{\underset{\sim}{r}} = \dfrac{\sqrt{18}}{3\underset{\sim}{i} - 3\underset{\sim}{j}}$

$\hat{\underset{\sim}{r}} = \dfrac{3\sqrt{2}}{3(\underset{\sim}{i} - \underset{\sim}{j})}$

$\hat{\underset{\sim}{r}} = \dfrac{\sqrt{2}}{\underset{\sim}{i} - \underset{\sim}{j}}$

9. a) The sum of the first n term of an arithmetic progression is given by $S_n = 72n - 2n^2$, find the ninth term. [2 marks]

b) Find the value of n for which the sum of the first 2n terms is 28 times the sum of first n terms of a geometric progression which has a common ratio of 3. [3 marks]

Answer :

(a) $T_9 = a + (n-1)d$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_n = \frac{n}{2}[2T_n]$

$2T_n = \frac{2S_n}{n}$

$T_n = \frac{S_n}{n}$

$\cancel{T_9 = \frac{72\,2\,2}{9}}$

$T_9 = \frac{72(9) - 2(9)^2}{9}$

$T_9 = 54$

(a) $T_9 = S_9 - S_8$

$= (72(9) - 2(9)^2) - (72(8) - 2(8)^2)$

$= 486 - 448$

$= 38$

(b) $r = 3$

$S_{2n} = 28\,S_n$

$\frac{a(3^{2n}-1)}{3-1} = 28\left(\frac{a(3^n-1)}{3-1}\right)$

$\frac{a(3^{2n}-1)}{2} = \overset{14}{28}\left(\frac{a(3^n-1)}{2}\right)$

$a(3^{2n}-1) = 28\,(a(3^n-1))$

$3^{2n} - 1 = 28\,(3^n - 1)$

~~$3^{2n} - 3^1 = (1+27)(3^n - 3^1)$~~

~~$3^{2n} - 3^1 = (3^1 + 3^3)(3^n - 3^1)$~~

~~$2n - 1 = (1+3)(n-1)$~~

~~$2n - 1 = n - 1 + 3n - 3$~~

~~$3 = 2n$~~

~~$n = \frac{3}{2}$~~

Let $x = 3^n$

$x^2 - 1 = 28(x - 1)$

$x^2 - 1 = 28x - 28$

$x^2 - 28x + 27 = 0$

$(x - 27)(x - 1) = 0$

$x - 27 = 0$, $x - 1 = 0$

$x = 27$ $x = 1$

$3^n = 27$ $3^n = 1$

$3^n = 3^3$ $3^n = 3^1$

$n = 3$ $n = 1$

$\therefore n = 3$ or $n = 1$

[illegible]

10. a) If $\sqrt{7}x = \sqrt{2}x + \sqrt{3}$, find the value of x in the form of $\frac{\sqrt{p}+\sqrt{6}}{q}$.

[2 marks]

b) Solve the equation $2\ln\frac{3x}{\sqrt{x}} - \ln(x+1) = 2$. Give your answer correct to 4 decimal places.

[3 marks]

Answer :

(a) $\sqrt{7}x = \sqrt{2}x + \sqrt{3}$

$\sqrt{7}x - \sqrt{2}x = \sqrt{3}$

$x(\sqrt{7}-\sqrt{2}) = \sqrt{3}$

$x = \frac{\sqrt{3}}{\sqrt{7}-\sqrt{2}}$

$\cancel{x = \sqrt{3}}$

$x = \frac{\sqrt{3}}{\sqrt{7}-\sqrt{2}} \times \frac{\sqrt{7}+\sqrt{2}}{\sqrt{7}+\sqrt{2}}$

$x = \frac{\sqrt{21}+\sqrt{6}}{7+\sqrt{14}-\sqrt{14}-2}$

$x = \frac{\sqrt{21}+\sqrt{6}}{5}$

(b) $2\ln\frac{3x}{\sqrt{x}} - \ln(x+1) = 2$

$\ln\left(2\left(\frac{3x}{\sqrt{x}}\right) - (x+1)\right) = 2$

$\frac{6x}{\sqrt{x}} - x - 1 = e^2$

$\frac{6x}{\sqrt{x}} - x - 1 = 7.389$

$\frac{6x - x\sqrt{x}}{\sqrt{x}} = 8.389$

(b) $2\ln\frac{3x}{\sqrt{x}} - \ln(x+1) = 2$

$\ln\left(\frac{9x^2}{x}\right) - \ln(x+1) = 2$

$\ln\left(\frac{9x}{x+1}\right) = 2$

$\frac{9x}{x+1} = e^2$

$9x = e^2(x+1)$

$9x = e^2x + e^2$

$9x - e^2x = e^2$

$x(9-e^2) = e^2$

$x = \frac{e^2}{9-e^2}$

11. a) Prove that ${}^{n+1}C_2 - {}^{n}C_2 = n.$ [2 marks]

b) A family consisting of a father, a mother and 5 children are seated at a round table. Find the number of different ways they can be seated if
i) there are no conditions.
ii) the father and the mother are seated together. [4 marks]

Answer :

(a) $${}^{n+1}C_2 - {}^{n}C_2 = \frac{(n+1)!}{2!(n+1-2)!} - \frac{n!}{2!(n-2)!}$$

$$= \frac{(n+1)!}{2!(n-1)!} - \frac{n!}{2!(n-2)!}$$

$$= \frac{(n+1)(n)\cancel{(n-1)!}}{2!\cancel{(n-1)!}} - \frac{n(n-1)\cancel{(n-2)!}}{2!\cancel{(n-2)!}}$$

$$= \frac{n^2+n}{2} - \frac{n^2-n}{2}$$

$$= \frac{n^2+n-n^2+n}{2}$$

$$= \frac{2n}{2}$$

$$= n \text{ (proven)}$$

(b)(i)
number of ways
$= (7-1)!$
$= 6!$
$= 720$

(ii) number of ways
$= (6-1)! \times 2!$
$= 5! \times 2!$
$= 240$

12.

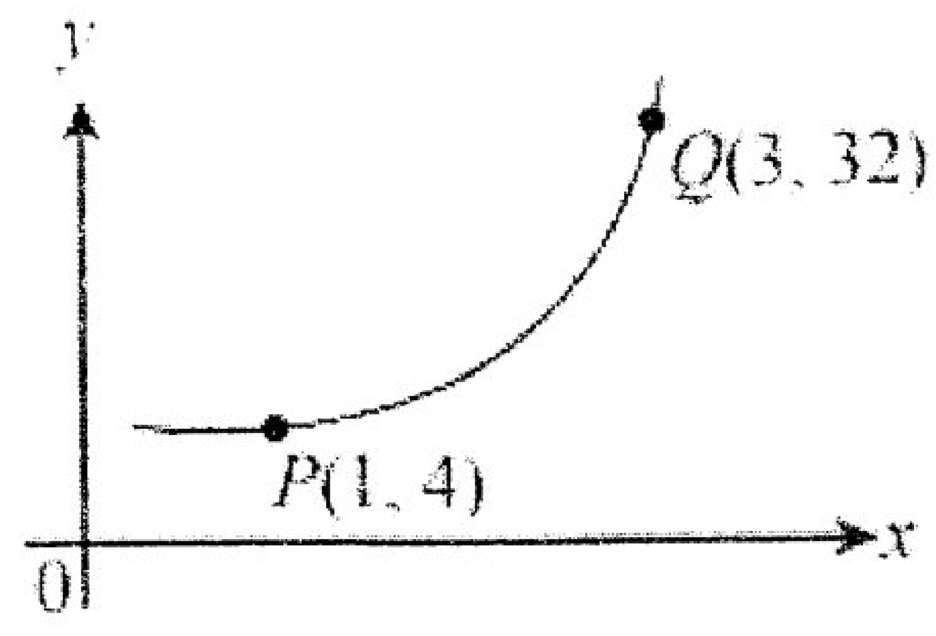


Diagram 12

$32 = ab$

Diagram 12 shows part of the graph of y against x for the equation $y = ab^x$, such that a and b are constants.

a) Sketch the straight line graph of $log_2\ y\ against\ x$. Mark and state the coordinates of the corresponding point P and Q. [2 marks]

b) Based on the graph in (a), find the value of $a\ and\ b$. [4 marks]

Answer :

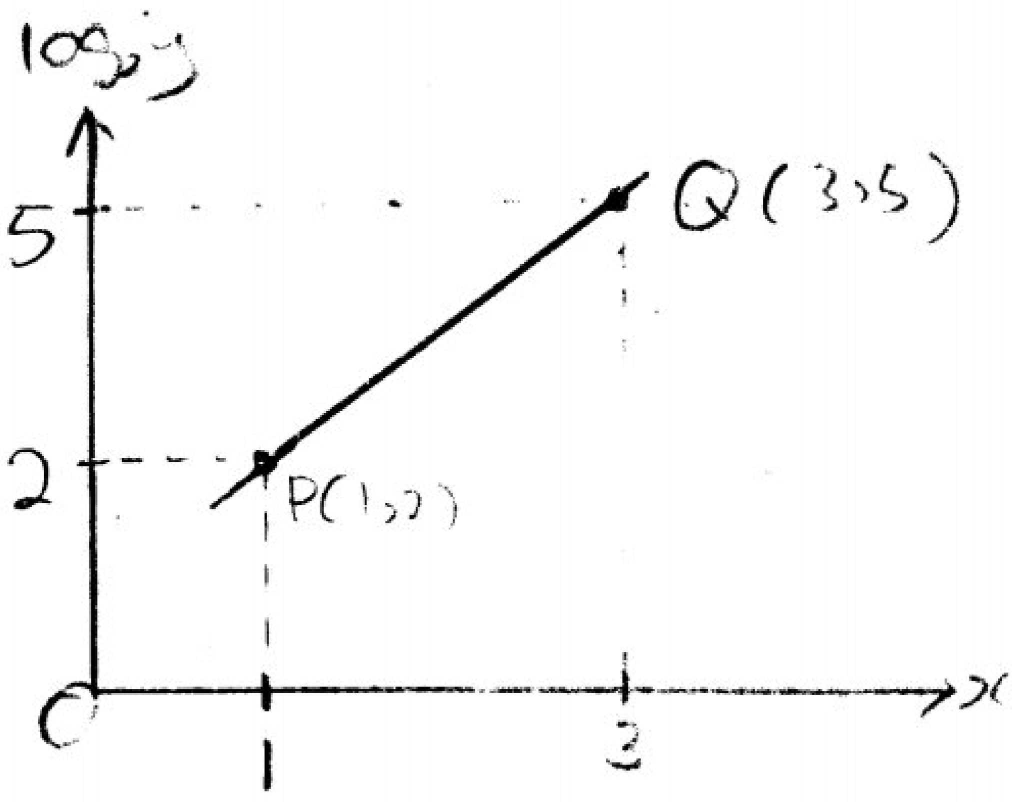
$\cancel{(a)\ y = ab^x}$

$\cancel{\log_2 y = \log_2 (ab^x)}$

$\cancel{\log_2 y = \log_2 a + \log_2 b^x}$

$\cancel{\log_2 y = (\log_2 b)x + \log_2 a}$

(a)



(b) $y = ab^x$

$\log_2 y = \log_2 (ab^x)$

$\log_2 y = \log_2 a + \log_2 b^x$

$\log_2 y = (\log_2 b)x + \log_2 a$

$\log_2 b = m$

$\log_2 b = \frac{5-2}{3-1}$

$\log_2 b = \frac{3}{2}$

$b = 2^{\frac{3}{2}}$

$b = 2.848$

$\log_2 y = \frac{3}{2}x + \log_2 a$

$5 = \frac{3}{2}(3) + \log_2 a$

$\log_2 a = 5 - \frac{9}{2}$

$\log_2 a = \frac{1}{2}$

$a = 2^{\frac{1}{2}}$

$a = 1.414$

Section B

[16 marks]

Answer **two** questions.

13. a) Without using calculator, find the value of $\tan 15^o$ in the form $p \pm \sqrt{q}$, where p and q are integers. [3 marks]

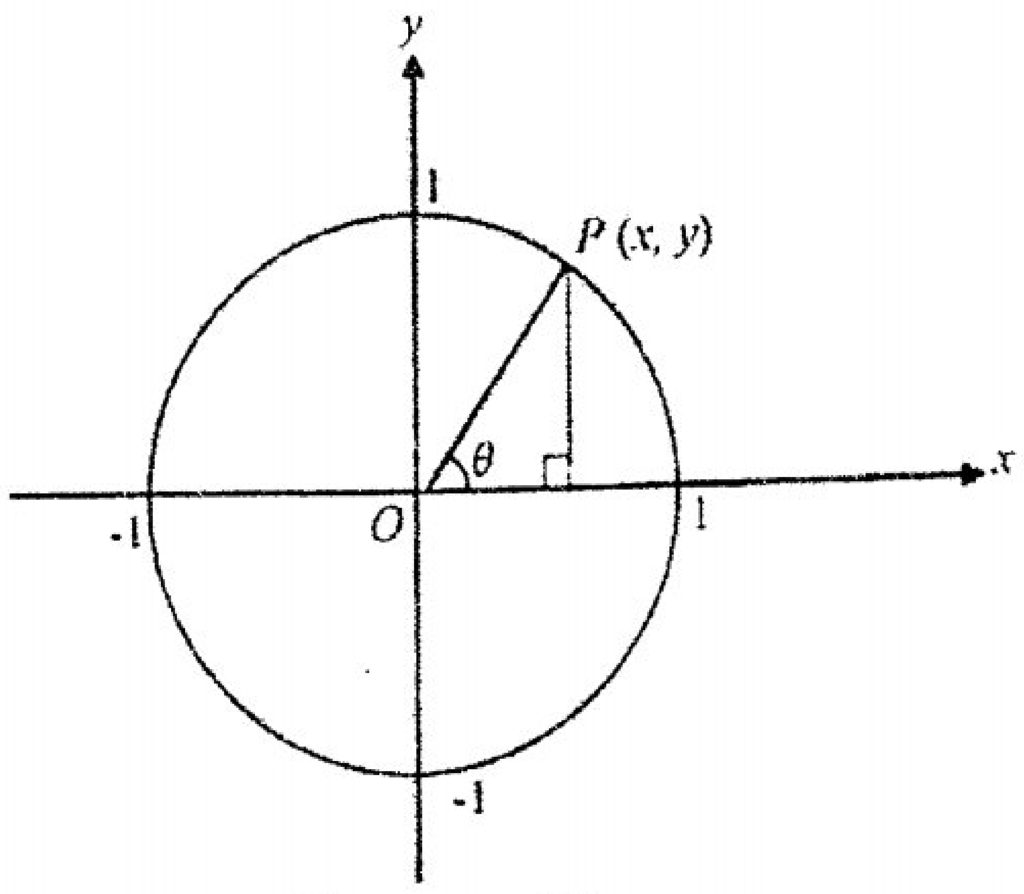


Diagram 13

b) Diagram 13 shows a unit circle in the Cartesian plane.
Given $x = \cos\theta$ *and* $y = \sin\theta$,

i) derive a basic identity involving $\cos\theta$ *and* $\sin\theta$. [2 marks]

ii) Hence, solve the trigonometric equation
$2\cos^2\theta - 5 = 7\sin\theta$ *for* $0^o \le \theta \le 360^o$. [3 marks]

Answer :

(a) $\tan 15^\circ = -\tan(90^\circ - 15^\circ)$

$= -\tan(75^\circ)$

$= -\tan(45^\circ + 30^\circ)$

$= -\tan 45^\circ - \tan 30^\circ$

$= -1 - \frac{1}{\sqrt{3}}$

$= -1 - \left(\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\right)$

$= -1 - \left(\frac{\sqrt{3}}{3}\right)$

$= \frac{-3-\sqrt{3}}{3}$

14.

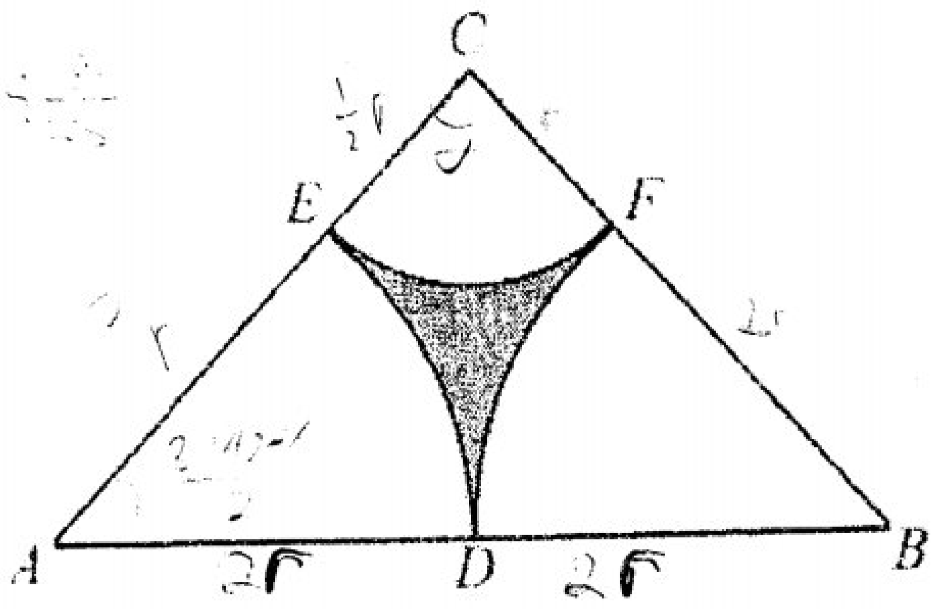


Diagram 14

Diagram 14 shows 3 sectors of circles with centre A, B and C respectively, touching each other at point D, E and F.
Given $AD = DB = 2r$, $AE = 2EC$, $< ACB = \alpha\ radian$ and the perimeter of shaded region is $9\frac{81}{125}$ unit.

a) Express r in terms of α and π. [3 marks]

b) Calculate the area of the shaded region if r = 2. [5 marks]

Answer :

(a) AE = r

$EC = \frac{1}{2}r$

Perimeter of shaded region $= 9\frac{81}{125}$

$r\left(\frac{\pi - \alpha}{2}\right) + r\left(\frac{\pi - \alpha}{2}\right) + \frac{1}{2}r\alpha = 9\frac{81}{125}$

$2\left[r\left(\frac{\pi - \alpha}{2}\right)\right] + \frac{1}{2}r\alpha = 9\frac{81}{125}$

$r(\pi - \alpha) + \frac{1}{2}r\alpha = 9\frac{81}{125}$

$r\left(\pi - \alpha + \frac{1}{2}\alpha\right) = 9\frac{81}{125}$

$r\left(\pi - \frac{1}{2}\alpha\right) = 9\frac{81}{125}$

$r = 9\frac{81}{125(\pi - \frac{1}{2}\alpha)}$

~~c) Area of shaded region~~

$= \frac{1}{2}(\sqrt{\ }$

(b)

$CD = \sqrt{\left(\frac{3}{2}(2)\right)^2 - (2)^2}$

$= \sqrt{5}$

$\tan\left(\frac{1}{2}\alpha\right) = \frac{2}{\sqrt{5}}$

$\frac{1}{2}\alpha = \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$

$\alpha = 2 \times 41.81$

$\alpha = 83.62 \times \frac{3.142}{180}$

$\alpha = 1.460$

Area of shaded region

$= \frac{1}{2}(2+2)(\sqrt{5}) - 2\left(\frac{1}{2} \times 2^2 \times \frac{3.142 - 1.460}{2}\right)$

$- \frac{1}{2}\left(\frac{1}{2}(2)\right)^2(1.460)$

$= 2\sqrt{5} - 3.364 - 0.73$

$= 0.3781$ units^2

15.

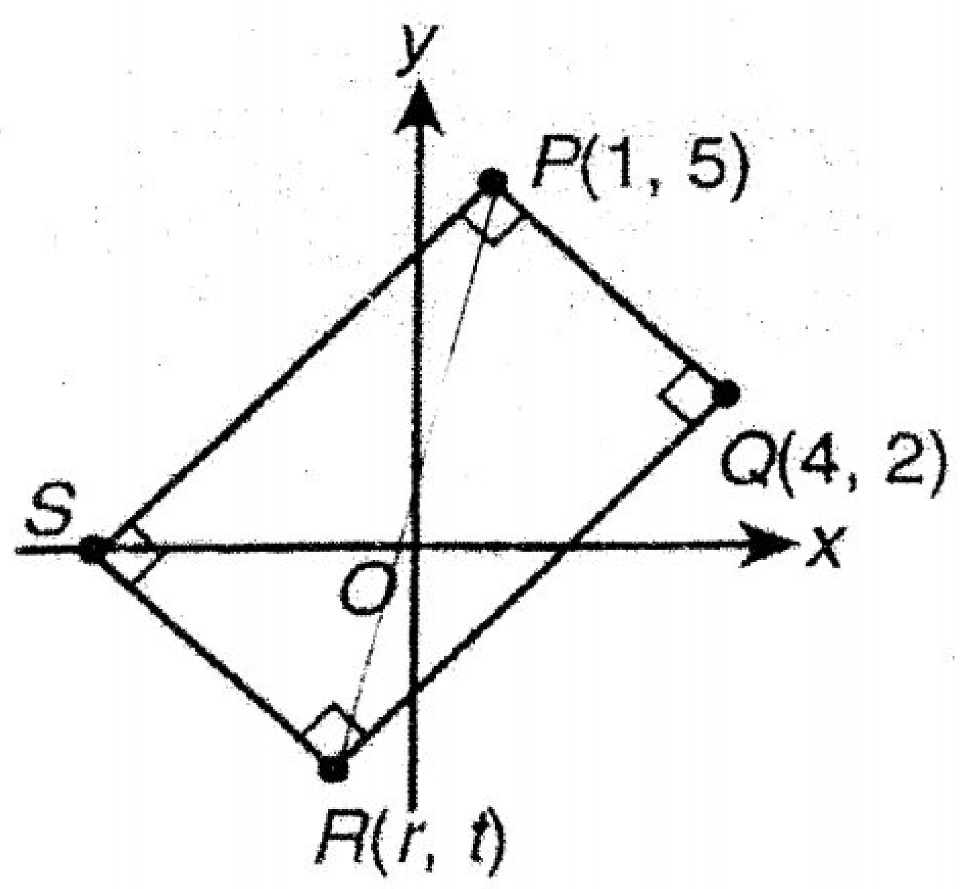


Diagram 15

Diagram 15 shows the vertices of the rectangle PQRS on the Cartesian plane.

a) Find the equation which relates r and t. [2 marks]

b) Show that the area of triangle PQR is $9-\frac{3}{2}(r+t)$. [3 marks]

c) If the area of the rectangle PQRS is 30 $unit^2$, find the coordinates of point R. [3 marks]

Answer :

15(a) $m_{PQ}=\frac{5-2}{1-4}$

$=-1$

$m_{RS}=m_{PQ}$

$=-1$

Equation :

$y-t=-1(x-r)$

$y=-x+r+t$

(b) ~~S(x,0)~~

~~$0=-x+r+t$~~

~~$x=r+t$~~

~~S(r+t,0)~~

(b) Area of triangle PQR

$=\frac{1}{2}\begin{vmatrix}1 & 4 & r & 1\\ 5 & 2 & t & 5\end{vmatrix}$

$=\frac{1}{2}|(1\times2+4\times t+r\times5)-(5\times4+2\times r+t\times1)|$

$=\frac{1}{2}|(2+4t+5r)-(20+2r+t)|$

$=\frac{1}{2}|2+4t+5r-20-2r-t|$

$=\frac{1}{2}|3t+3r-18|$

$=-\frac{18}{2}+\frac{3t+3r}{2}$

$=-9+\frac{3(t+r)}{2}$

$=-9+\frac{3}{2}(t+r)$

$\times -1$:

$=9-\frac{3}{2}(r+t)$

(shown)

(c) Area of rectangle PQRS = 30

$2(9-\frac{3}{2}(r+t))=30$

$9-\frac{3}{2}(r+t)=15$

$\frac{3}{2}(r+t)=-6$

$r+t=-4$ — 1

S(x,0)

$y=-x+r+t$

$0=-x+r+t$

$x=r+t$

$x=-4$

S(-4,0)

~~$\frac{t-0}{r-(-4)}=-1$~~

~~$\frac{t}{r+4}=-1$~~

~~$t=-1(r+4)$~~

~~$t=-r-4$ — 2~~

~~sub 2 into 1:~~

~~$r-r-4=4$~~

$1-4=-4-r$

$r=-1$

sub $r=-1$ into 1 :

$r+t=4$

$t=-4+1$

$t=-3$

$\therefore R(-1,-3)$

15(a)

$\left(\frac{5-2}{1-4}\right)\left(\frac{2-t}{4-r}\right)=-1$

$-1\left(\frac{2-t}{4-r}\right)=-1$

$\frac{2-t}{4-r}=1$

$2-t=4-r$

$r=2+t$

END OF QUESTION PAPER